

Performance of ML Range Estimator in Radio Interferometric Positioning Systems

Yue Zhang, Wangdong Qi*, *Member, IEEE*, Guangxia Li, Su Zhang

Abstract

The radio interferometric positioning system (RIPS) is a novel positioning solution used in wireless sensor networks. This letter explores the ranging accuracy of RIPS in two configurations. In the linear step-frequency (LSF) configuration, we derive the mean square error (MSE) of the maximum likelihood (ML) estimator. In the random step-frequency (RSF) configuration, we introduce average MSE to characterize the performance of the ML estimator. The simulation results fit well with theoretical analysis. It is revealed that RSF is superior to LSF in that the former is more robust in a jamming environment with similar ranging accuracy.

Index Terms

Radio interferometric positioning system, maximum likelihood estimator, method of interval error, outlier probability, average MSE, average ambiguity function.

I. INTRODUCTION

The radio interferometric positioning system (RIPS), a node localization system used in wireless sensor networks, has received significant attention in recent years [1], [2], [3], [4], [5]. The novel ranging scheme introduced in RIPS is key to its success in providing low-cost and accurate localization solutions. However, our knowledge of the ranging performance of RIPS is rather limited because of the lack of systematic investigation. This letter examines its performance in two measurement configurations including linear step-frequency (LSF) and random step-frequency (RSF). LSF is applied in mobile node tracking and

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The authors are with the PLA University of Science and Technology, Nanjing, Jiangsu 210007, China (e-mail: zhyemf@gmail.com; wangdongqi@gmail.com; 13905177686@139.com; franklinzhang1985@gmail.com).

landslide early warning systems [4], [5] whereas RSF can be essential for military applications with inherent anti-jamming capabilities [6].

The topic of interest here is the ranging accuracy of RIPS both in high signal-to-noise ratio (SNR) and moderate-low SNR since RIPS may be deployed in a variety of environments. Because the performance predictions provided by lower bounds such as the Cramer-Rao bound (CRB) and the Ziv-Zakai bound (ZZB) are too optimistic when the SNR is below a certain threshold [7], [8], we choose to characterize the ranging performance of RIPS by using the mean square error (MSE) of the maximum likelihood (ML) estimator.

To obtain the MSE of the ML estimator in the entire SNR region, we employ the method of interval error (MIE) [7], [8]. In the LSF configuration, the measurement frequencies are fixed so we can obtain the MSE with MIE directly. In the RSF configuration, because the MSE is a random variable with respect to hopping frequencies in measurement, we use the average MSE (AMSE) [9] to characterize the performance of the ML estimator. We introduce the average ambiguity function (AAF) [10] to facilitate the derivation of AMSE to avoid the tedious process of averaging MSEs under different measurement frequencies.

The theoretical results are verified by simulations. The ranging accuracy of RSF is shown to be very similar to that of LSF. Therefore, RSF is superior to LSF for military applications because it is more robust in a jamming environment.

II. SYSTEM MODEL

The basic unit of a ranging process in RIPS involves two nodes, A and B, simultaneously emitting a pair of sine waves at two close frequencies with a difference of δ , whereas other two nodes, C and D, measure the phase of the beat signal of the two sine waves. The overall ranging process consists of multiple such units at a series of frequency pairs. According to [1], the phase offset φ_i between C and D of the i th beat signal is related to the so-called *qrange* $d_0 = d_{AD} - d_{AC} + d_{BC} - d_{BD}$ (d_{XY} is the distance between node X and Y) as

$$\varphi_i = \left(2\pi \frac{f_i}{c} d_0 + \theta + n_i \right) \mod 2\pi, \quad (1)$$

where f_i is the average of the i th pair of frequencies ($i = 1, \dots, M$), c is the speed of signal propagation, n_i is independent and identically distributed (i.i.d) Gaussian white noise with variance σ^2 , and $\theta = \frac{2\pi\delta}{c}(d_{AD} - d_{AC} - d_{BC} + d_{BD})$ is a constant related to d_0 . We define the SNR as $1/\sigma^2$.

Essentially, the ranging process in RIPS distills to a parameter estimation problem where the qrange d_0 , a linear combination of distances between the four nodes, is determined according to the observation of the phase differences $\Phi = \{\varphi_1, \dots, \varphi_M\}$ of the beat signals on measurement frequencies $\mathbf{f} = \{f_1, \dots, f_M\}$.

It should be noted here that qranges can be used with ease in the localization process in similar ways like distances. Due to space limitations, we refer the readers to [4] for further information.

The observation equation (1) has been simplified in previous study by neglecting the term θ [1]. In this letter, we retain the general form of (1) by treating θ as an unknown parameter to accommodate additional scenarios.

We assume that all measurement frequencies employed in the ranging process are multiples of the system's minimum frequency interval f_{min}

$$f_i = (k_0 + k_i)f_{min}, \quad (2)$$

where $k_0 f_{min}$ is the initial frequency. Assuming that measurement frequencies are chosen from the available bandwidth B in RIPS, it is clear that the total number of measurement frequencies is $N = B/f_{min} + 1$.

Obviously, the ranging process in RIPS is defined by the configuration of measurement frequencies \mathbf{f} . In the LSF configuration, f_i proceeds in a constant step, i.e., $k_i = (i - 1)\frac{N-1}{M-1}$. We assume that $N - 1$ can be divided by $M - 1$ for convenience. In the RSF configuration, the M measurement frequencies are chosen randomly from all available frequencies so that the positive integers k_i are random variables distributed uniformly in $[0, N - 1]$.

III. PERFORMANCE ANALYSIS

We first present the ML estimator of qrange in RIPS. In a fairly large SNR region, the observation equation (1) can be converted into an equivalent form [11], [12]:

$$\exp(j\varphi_i) = \exp\left(j\left(2\pi\frac{f_i}{c}d_0 + \theta\right)\right) + z_i, \quad (3)$$

where z_i is i.i.d complex Gaussian white noise with variance $2\sigma^2$ corresponding to the additive phase noise n_i with variance σ^2 in (1). The estimation of qrange d_0 is equivalent to single-tone frequency estimation if we regard d_0 as the frequency of a single tone and f_i/c as the discrete sample time. Then the joint distribution function of $\exp(j\Phi)$ at \mathbf{f} with the unknown parameter vector $\mathbf{A} = [d_0, \theta]^T$ is [13]

$$f(\exp(j\Phi); \mathbf{A}) = \left(\frac{1}{2\pi\sigma^2}\right)^M \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^M (a_i - \mu_i)^2 + \sum_{i=1}^M (b_i - v_i)^2\right)\right], \quad (4)$$

where $a_i = \mathbf{Re}(\exp j\varphi_i)$, $b_i = \mathbf{Im}(\exp j\varphi_i)$, $\mu_i = \cos(2\pi \frac{f_i}{c}d_0 + \theta)$, and $v_i = \sin(2\pi \frac{f_i}{c}d_0 + \theta)$. As a result, the ML estimate \hat{d}_0 is obtained by maximizing the objective searching function (OSF)

$$V(d) = \left| \sum_{i=1}^M \left\{ \exp(j\varphi_i) \exp(-j2\pi \frac{f_i}{c}d) \right\} \right|, \quad (5)$$

where $d \in [d_0 - \frac{d_{max}}{2}, d_0 + \frac{d_{max}}{2}]$, and d_{max} is the range of interest within the unambiguous distance of RIPS.

According to MIE [7], [8], we represent the MSE of the ML grange estimator in configuration \mathbf{f} as a weighted sum of the local error term and the global error term (outlier)

$$MSE(d_0|\mathbf{f}) = P_o \cdot E[(\hat{d}_0 - d_0)^2|outlier] + (1 - P_o) \cdot CRB(d_0|\mathbf{f}), \quad (6)$$

where the weights are given by the outlier probability P_o and the local error is approximated by the CRB term $CRB(d_0|\mathbf{f})$.

Next, we handle LSF and RSF in sections III-A and III-B.

A. The LSF Configuration

For the LSF configuration, (6) can be simplified further if we introduce the concept of the ambiguity function (AF), which is the OSF when the data in (5) is noise free. According to [8], an outlier is an event that occurs when the ML parameter estimate is outside the mainlobe of the AF. The AF can be discretized at the sidelobe peaks $d_n (n = 0, 1, \dots, N_p)$, where d_n s are positions and N_p is the number of sidelobe peaks of the AF. Under this discretization, P_o and the first term in (6) can be simplified as [7]

$$P_o \approx \sum_{n=1}^{N_p} p_n \quad (7)$$

and

$$P_o \cdot E[(\hat{d}_0 - d_0)^2|outlier] \approx \sum_{n=1}^{N_p} p_n (d_n - d_0)^2, \quad (8)$$

where $p_n = \Pr[V(d_0) < V(d_n)]$ is the probability that the sidelobe peak of OSF at d_n is higher than the mainlobe.

Combining (6), (7) and (8), the MSE of the ML grange estimation in LSF can be approximated as

$$MSE_{LSF} \approx \sum_{n=1}^{N_p} p_n (d_n - d_0)^2 + \left(1 - \sum_{n=1}^{N_p} p_n \right) CRB_{LSF}, \quad (9)$$

where CRB_{LSF} represents $CRB(d_0|\mathbf{f})$ in the LSF configuration. We now address the determination of terms in (9).

1) *CRB*: The elements of the Fisher information matrix \mathbf{J} corresponding to (4) can be written as

$$\mathbf{J}_{ij} = -E \left[\frac{\partial^2 \ln f(\exp(j\Phi); \mathbf{A})}{\partial A_i \partial A_j} \right]. \quad (10)$$

Inverting \mathbf{J} yields CRB for the ML estimator of d_0 such that

$$CRB(d_0|\mathbf{f}) = \mathbf{J}_{11}^{-1} = \frac{Mc^2\sigma^2}{4\pi^2} \left/ \left[M \sum_{i=1}^M f_i^2 - \left(\sum_{i=1}^M f_i \right)^2 \right] \right. \quad (11)$$

Replacing f_i with the right hand side of (2), we get

$$CRB(d_0|\mathbf{f}) = \frac{Mc^2\sigma^2}{4\pi^2 f_{min}^2} \frac{1}{\mathbf{K}^T \mathbf{W} \mathbf{K}}, \quad (12)$$

where $\mathbf{K} = [k_1, \dots, k_M]^T$, \mathbf{W} is an $M \times M$ symmetric matrix with main diagonal elements $M - 1$ and others -1 .

Considering that k_i increases stepwise by $\frac{N-1}{M-1}$ in the LSF configuration, we get from (12)

$$CRB_{LSF} = \frac{3c^2\sigma^2(M-1)}{\pi^2 B^2 M(M+1)}. \quad (13)$$

2) *Outlier Related Terms*: From the ambiguity function

$$\begin{aligned} G(d) &= \left| \sum_{i=1}^M \exp \left(j2\pi \frac{f_i}{c} (d_0 - d) \right) \right| \\ &= \left| \frac{\sin \left(\pi (d_0 - d) B \frac{M}{(M-1)c} \right)}{\sin \left(\pi (d_0 - d) \frac{1}{(M-1)c} \right)} \right|, \end{aligned} \quad (14)$$

we get $N_p = M - 2$ and $d_n = d_0 + (-1)^n \frac{(M-1)c}{MB} (\lceil N/2 \rceil + 0.5)$.

Let

$$\begin{aligned} y_0 &= \sum_{i=1}^M \exp(jn_i) \\ y_n &= \sum_{i=1}^M \exp \left(j2\pi \frac{f_i}{c} (d_0 - d_n) + jn_i \right); \end{aligned} \quad (15)$$

we have $V(d_0) = |y_0|$, $V(d_n) = |y_n|$, and

$$p_n = \Pr(|y_0| - |y_n| < 0) = \Pr(|y_0|^2 - |y_n|^2 < 0). \quad (16)$$

It is observed that y_0 , as well as y_n , is the sum of M i.i.d random variables. In view of central-limit theorem, both y_0 and y_n are approximately Gaussian distributed if M is sufficiently large. In addition, because y_0 and y_n are correlated, we can get the expression of p_n resorting to appendix B in [14] by means of the first- and second-order moments of y_0 and y_n .

It should be noted that if x is normally distributed, we have $E[\exp(jx)] = e^{-\sigma^2/2}$ according to the definition of the character function [15]. Then, routine computation produces the first- and second-order

moments of y_0 and y_n :

$$\begin{aligned} E[y_0] &= M e^{-\sigma^2/2} \\ E[y_n] &= M r_n e^{-\sigma^2/2} \\ \text{var}[y_0] &= \text{var}[y_n] = M(1 - e^{-\sigma^2}) \\ \text{cov}[y_0, y_n] &= M(r_n)^*(1 - e^{-\sigma^2}), \end{aligned} \quad (17)$$

where $r_n = \frac{1}{M} \sum_{i=1}^M \exp\left(j2\pi \frac{f_i}{c}(d_0 - d_n)\right)$ is the relative sidelobe level of the n th sidelobe of the ambiguity function [7], and the superscript $(\cdot)^*$ means conjugation.

Substituting (17) into B-21 of [14], we have

$$p_n = Q_1(a, b) - \frac{1}{2} I_0(ab) \exp\left[-\frac{1}{2}(a^2 + b^2)\right]. \quad (18)$$

Here, $Q_1(\cdot, \cdot)$ is Marcum's Q function, $I_0(\cdot)$ is the modified Bessel function of the first kind and order 0, and $a = \sqrt{\frac{M}{2(e^{\sigma^2}-1)}} \left(1 - \sqrt{1 - |r_n|^2}\right)$, $b = \sqrt{\frac{M}{2(e^{\sigma^2}-1)}} \left(1 + \sqrt{1 - |r_n|^2}\right)$.

By now, all of the unknown terms in (9) have been determined, and we finally have a closed-form expression of the MSE in the LSF configuration.

B. The RSF Configuration

The $MSE(d_0|\mathbf{f})$ in (6) is a random variable in RSF because \mathbf{f} is a random vector. We choose to characterize the ranging performance of RIPS in the RSF configuration with the average of $MSE(d_0|\mathbf{f})$

$$\overline{MSE}_{RSF} = \bar{P}_o \cdot E[(\hat{d}_0 - d_0)^2 | \text{outlier}], + (1 - \bar{P}_o) \cdot \overline{CRB}_{RSF}, \quad (19)$$

where \overline{MSE}_{RSF} , \overline{CRB}_{RSF} , and \bar{P}_o are the averages of $MSE(d_0|\mathbf{f})$, $CRB(d_0|\mathbf{f})$, and the outlier probability with respect to the random vector \mathbf{f} .

Rather than obtaining \bar{P}_o by the traditional method in which P_o s with different realizations of \mathbf{f} are calculated one by one, we obtain the expression of \bar{P}_o immediately with the help of a concept known as the AAF, which is commonly used in random signal radars [10].

Averaging (5) with respect to \mathbf{f} and replacing φ_i by (1) with noise free data, we get the AAF of RSF

$$\begin{aligned} \bar{G}(d) &= E \left| \sum_{i=1}^M \exp\left(j2\pi \frac{f_i}{c}(d_0 - d)\right) \right| \\ &= \left| \frac{M \sin(\pi(d_0 - d) \frac{N f_{min}}{c})}{N \sin(\pi(d_0 - d) \frac{f_{min}}{c})} \right|. \end{aligned} \quad (20)$$

Similar to the case in LSF, we have

$$\bar{P}_o \approx \sum_{n=1}^{N_q} q_n \quad (21)$$

and

$$\bar{P}_o \cdot E[(\hat{d}_0 - d_0)^2 | outlier] \approx \sum_{n=1}^{N_q} q_n (d'_n - d_0)^2, \quad (22)$$

where d'_n are positions, N_q is the number of sidelobe peaks of the AAF, and $q_n = \Pr[V(d_0) < V(d'_n)]$ is the probability that the sidelobe peak of OSF at d'_n is higher than the mainlobe.

It follows from (20) that $N_q = N - 2$ and $d'_n = d_0 + (-1)^n \frac{c}{N f_{min}} (\lceil N/2 \rceil + 0.5)$.

Combining (19), (21), and (22), we have the closed-form expression of AMSE for RSF

$$\overline{MSE}_{RSF} \approx \sum_{n=1}^{N_q} q_n (d'_n - d_0)^2 + \left(1 - \sum_{n=1}^{N_q} q_n\right) \overline{CRB}_{RSF}, \quad (23)$$

where \overline{CRB}_{RSF} and q_n will be determined in the following subsections.

1) *Average CRB*: Denoting $X = \mathbf{K}^T \mathbf{W} \mathbf{K}$ and $g(X) = 1/X$, we have

$$\overline{CRB}_{RSF} = E \left[\frac{Mc^2 \sigma^2}{4\pi^2 f_{min}^2} \frac{1}{X} \right] = \frac{Mc^2 \sigma^2}{4\pi^2 f_{min}^2} E[g(X)]. \quad (24)$$

The determination of $E[g(X)]$ involves the joint distribution function of the *quadratic form* $\mathbf{K}^T \mathbf{W} \mathbf{K}$, which is highly complex for the uniform distributed variables k_i [16]. We resort to approximations here.

Let η be the mean and ρ be the second-order moment of X . Expanding $g(X)$ into polynomials near η and retaining the first three terms, we have $g(X) \approx g(\eta) + g'(\eta)(X - \eta) + g''(\eta) \frac{(X - \eta)^2}{2}$. Thus $E[g(X)]$ can be approximated as

$$E[g(X)] \approx g(\eta) + \frac{g''(\eta)}{2} E[(X - \eta)^2] = \frac{\rho}{\eta^3}. \quad (25)$$

To obtain η and ρ , the different orders of moment of k_i should be determined first. Considering that N is a very large number because the minimum frequency interval f_{min} can be as small as 1 Hz in modern transceivers [17], the a th-order moment of k_i can be expressed as

$$E(k_i^a) = \frac{1}{N} \sum_{x=0}^{N-1} x^a \approx \frac{1}{N} \int_0^N x^a = \frac{N^a}{a+1}. \quad (26)$$

For the two forms of quadratic terms k_i^2 and $k_i k_j (i \neq j)$ in X , the expectations are $E(k_i^2)$ and $E(k_i)^2$, respectively. Because the sum of coefficients of the first form is $M^2 - M$ and that of the second form is $-(M^2 - M)$, we have

$$\begin{aligned} \eta &= (M^2 - M)E(k_i^2) - (M^2 - M)E(k_i)^2 \\ &= \frac{M(M-1)N^2}{12}. \end{aligned} \quad (27)$$

Similarly, ρ is the sum of expectations of various quartic terms. These expectations have five forms: $E(k_i^4)$, $E(k_i)E(k_j^3)$, $E(k_i^2)^2$, $E(k_i^2)E(k_j)^2$ and $E(k_i)^4$ where $i \neq j$. The sums of coefficients of these

five forms are $\beta_1 = M(M-1)^2$, $\beta_2 = -4M(M-1)^2$, $\beta_3 = M(M-1)[(M-1)^2 + 2]$, $\beta_4 = -2M(M-1)(M-2)(M-3)$, and $\beta_5 = M(M-1)(M-2)(M-3)$. Therefore,

$$\begin{aligned}\rho &= \beta_1 E(k_i^4) + \beta_2 E(k_i)E(k_j^3) + \beta_3 E(k_i^2)^2 \\ &\quad + \beta_4 E(k_i^2)E(k_i)^2 + \beta_5 E(k_i)^4 \\ &= \frac{M(M-1)(5M^2-M+6)N^4}{720}.\end{aligned}\tag{28}$$

Plugging (27) and (28) back into (24) and (25), we have

$$\overline{CRB}_{RSF} \approx \frac{3c^2\sigma^2}{5\pi^2 B^2} \frac{(5M^2 - M + 6)}{M(M-1)^2}.\tag{29}$$

2) *The Determination of q_n* : For RSF, the means and second-order moments of y_0 and y_n are

$$\begin{aligned}E[y_0] &= M e^{-\sigma^2/2} \\ E[y_n] &= M r_n e^{-\sigma^2/2} \\ \text{var}[y_0] &= 0.5M(1 - e^{-\sigma^2}) \\ \text{var}[y_n] &= 0.5M(1 - |r_n|^2 e^{-\sigma^2}) \\ \text{cov}[y_0, y_n] &= 0.5M(r_n)^*(1 - e^{-\sigma^2}),\end{aligned}\tag{30}$$

where $r_n = E \left[\exp \left(j2\pi \frac{f_i}{c} (d_0 - d'_n) \right) \right]$.

Substituting (30) into B-21 of [14], we have

$$q_n = Q_1(c, d) - v \cdot I_0(cd) \exp \left[-\frac{1}{2}(c^2 + d^2) \right],\tag{31}$$

where $c = \sqrt{\frac{M}{A_n+S} \left(2e^{\sigma^2} - A_n - \sqrt{A_n^2 + A_n S} \right)}$, $d = \sqrt{\frac{M}{A_n+S} \left(2e^{\sigma^2} - A_n + \sqrt{A_n^2 + A_n S} \right)}$, $v = \frac{\sqrt{1+S/A_n-1}}{2\sqrt{1+S/A_n}}$, of which $A_n = 1 - |r_n|^2$ and $S = 4(e^{2\sigma^2} - e^{\sigma^2})$.

Hence, we have obtained in (23) a closed-form expression of AMSE in the RSF configuration.

IV. SIMULATION RESULTS

In this section, the accuracy of the approximations derived in the previous section is verified through Monte Carlo simulations. For a fair comparison, we assume that LSF and RSF employ the same frequency band with a bandwidth of $B = 0$ MHz and the same minimum frequency interval $f_{min} = 1$ kHz. The number of Monte Carlo trials was 10^5 for each SNR. In each trial, we use M measurement frequencies for LSF and randomly choose M frequencies for RSF.

Fig.1 shows the MSE of the ML estimator as a function of SNR when M is 31. The MSE prediction is hard limited to never exceed the variance $d_{max}^2/12$ of an estimate assumed to be uniformly distributed over the search space.

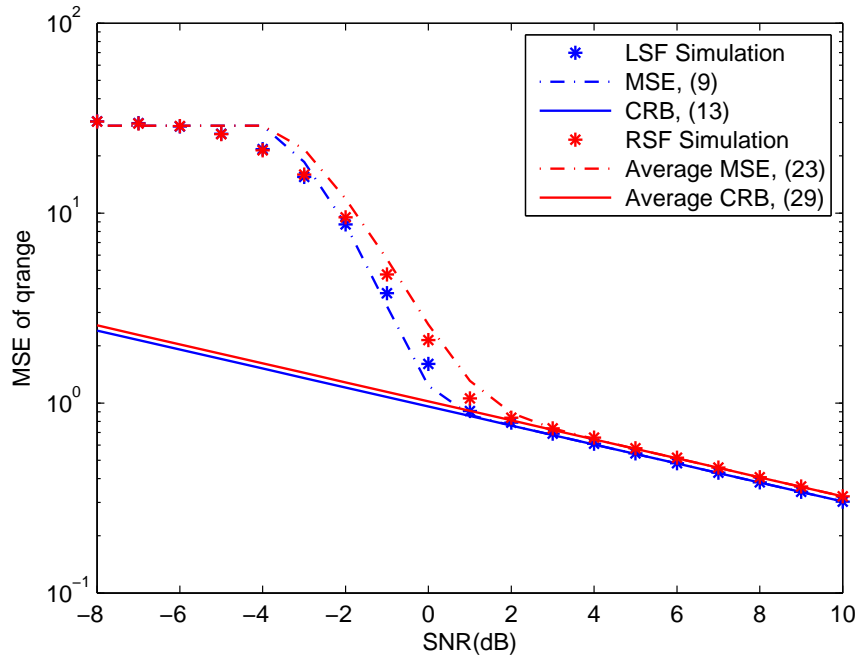


Fig. 1. Monte Carlo simulations of the ML range estimator in the LSF configuration and RSF configuration with $B = 30$ MHz and $M = 31$. The number of Monte Carlo trials was 10^5 . Blue: the MSE approximation of LSF is compared with simulation and CRB. Red: the AMSE approximation of RSF is compared with simulation and Average CRB.

We determined that both the derived MSE approximation in (9) for LSF and the AMSE approximation in (23) for RSF are accurate. Moreover, the AMSE in RSF is only slightly larger than the MSE in LSF in the entire SNR region. For military applications, RSF is superior to LSF because the tiny loss of accuracy in RSF compared with LSF is well compensated by its anti-jamming capabilities.

V. CONCLUSIONS

This letter provides closed-form expressions of the MSE of the ML range estimator for the RIPS in the LSF configuration and the AMSE in the RSF configuration. The simulation results agree well with the theoretical analysis. We conclude that RSF increases the anti-jamming capability of RIPS at a very small cost of slightly decreased ranging accuracy.

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